DG 2: Exploring Horizons of Knowledge for Teaching (Session 1)

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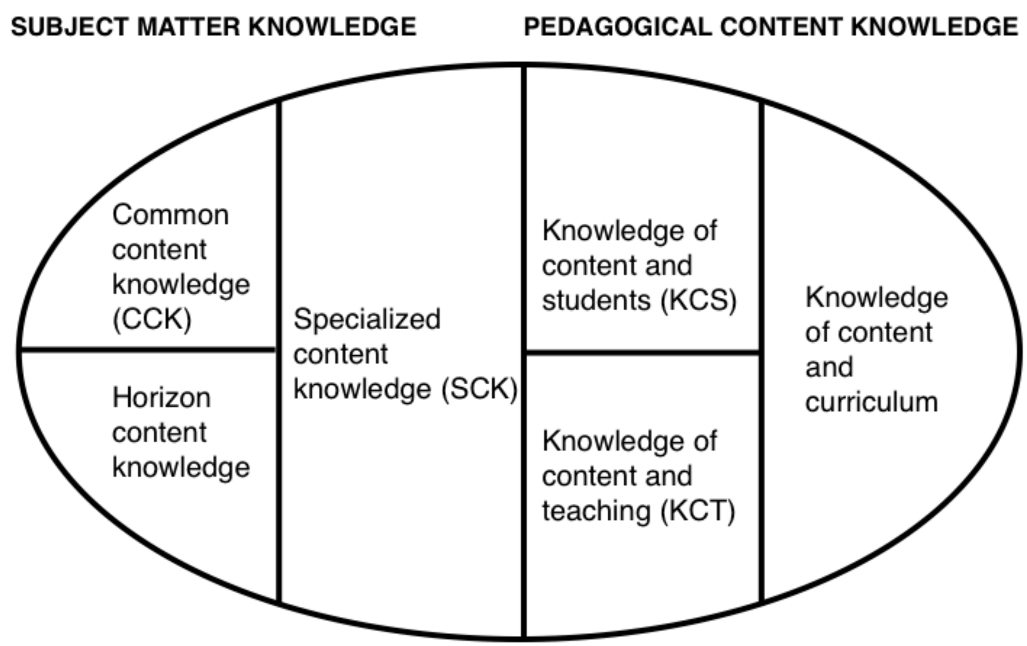
Introduction

- Mathematical Knowledge for Teaching (MKT), Ball, Thames, Phelps (2008)



Knowledge at the Mathematical Horizon

Ball and Bass (2009) identify four components:

1. A sense of the mathematical environment surrounding the current "location" in instruction.

2. Major disciplinary ideas and structures

3. Key mathematical practices

4. Core mathematical values and sensibilities

Motivation from Ball and Bass:

We think, however, that teaching can be more skillful when teachers have mathematical perspective on what lies in all directions, behind as well as ahead, for their pupils, that can serve to orient their navigation of the territory. We seek to work toward conceptualizing more precisely what comprises that sense of horizon a domain that involves a sense of how mathematics at play in instruction is related to the larger mathematical landscape (Ball & Bass, 2009).

Different approaches:

- Some researcher use Klein' s (1932) notion about elementary mathematics from an advanced standpoint.

- Others use the alternately: advanced mathematics from an elementary standpoint.

A practice-based approach for horizon content knowledge

Ribeiro & Jakobsen

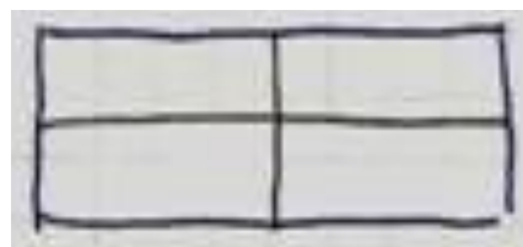
A practice-based approach for horizon content knowledge

- Ground analysis in the dynamics of instruction (Cohen, Raudenbush, & Ball, 2003; Ball & Bass, 2003; Ball & Forzani, 2007)
- Documentation of teaching, interviews (individual and focus group), questionnaires, professional development context
- Identify candidate teaching episode
  - Analyze work of teaching
  - Identify potential resources from advanced mathematics
- **Develop and review vignettes**
- Analysis across vignettes to build a definition

- Draw a rectangle and divide it into four equal parts – if possible present a diversity of answers, in the same sequence you think of.

Vignette of HCK

Maria intends to work with her year 2 students some aspects related with rectangles and fractions as part-whole. In that sense she prepared a task asking their students to draw a rectangle and divide it into four equal parts, and then to draw another rectangle and divide it into four equal parts in a different way.Two students draw the following:



Vignette of HCK (cont.)

When the teacher asked for something different, Maria draws:

And the following dialog occurs:

Maria: "I know the pieces don't look equal, but they're supposed to be — you can just push the lines in so the pieces are the same."

Jerome disagrees: "the middle ones are too big and the corner ones are too small."

Maria counters: "That's why you push the lines in. I didn't draw it right."

Alicia looks skeptical: "but you can't do that. That can't make them the same."

Work of teaching in vignette and HCK

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Mathematical work of teaching

Mathematical resources

Work of teaching in vignette and HCK

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Mathematical work of teaching:

- Hearing Maria’s thinking and Jerome’s and Alicia’s objections
- Is Maria’s idea reasonable? Worth pursuing?
- Talking about the problem and Maria’s idea with mathematical integrity

Work of teaching in vignette and HCK

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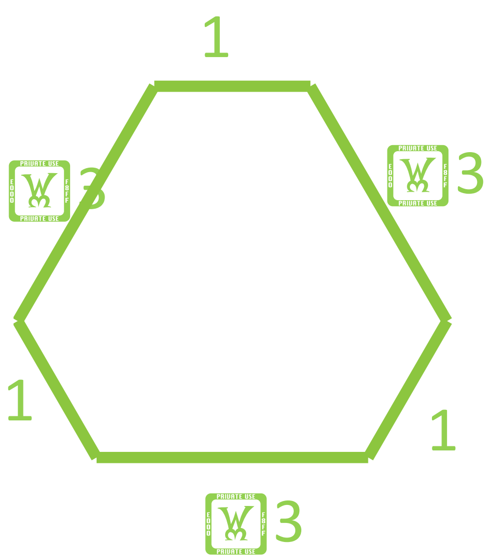
Mathematical resources:

- Recognition of Maria's argument as an application of the intermediate value theorem can help one hear the mathematical "reasonableness" of Maria's thinking
- An intuitive notion of continuity as tracing without lifting your pencil can provide an image for navigating a discussion about "pushing a line"
- Language to talk casually yet with integrity can be used to draw student attention to important mathematical ideas

Horizons, hexagons, and heed

Mamolo

Imagine the scenario…

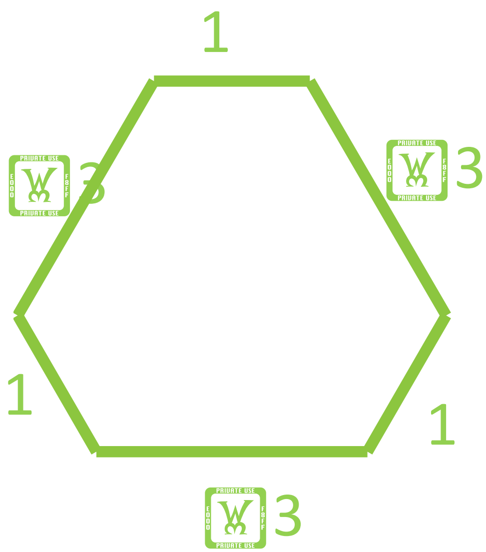


Delia's Hexagon

- Delia: a "good student" working on an extra-curricular problem and is having trouble determining the area of the depicted hexagon.

How do you recommend Delia go about finding the area?

Research Questions:



Delia's Hexagon

- What are pre-service secondary mathematics teachers' preferences when considering recommendations for how to determine the area of an irregular hexagon?
- What are the bases for these preferences?

Knowledge at the Mathematical Horizon

KMH is interpreted as a facet of teachers' subject matter knowledge. Ball and Bass (2009) identify four components:

1. A sense of the mathematical environment surrounding the current "location" in instruction.

Inner horizon

2. Major disciplinary ideas and structures

3. Key mathematical practices

Outer horizon

4. Core mathematical values and sensibilities

Z & M (2011)
Teachers' horizon

Adapting Husserl's "horizons"

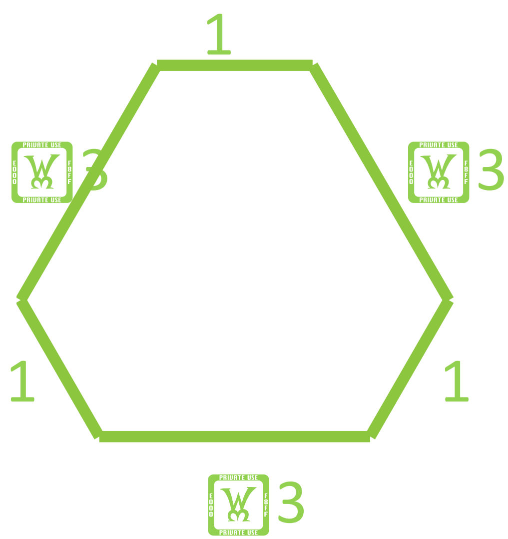
Inner Horizon: corresponds to aspects of an object that are not at the focus of attention but that are also intended

Outer Horizon: includes features which are not in themselves aspects of the object, but which are connected to the world in which the object exists (Follesdal, 1998, 2003).

Can be applied to analyse what is in and out of focus for individuals and what connections they make

Applying Husserl's "horizons"

Example:

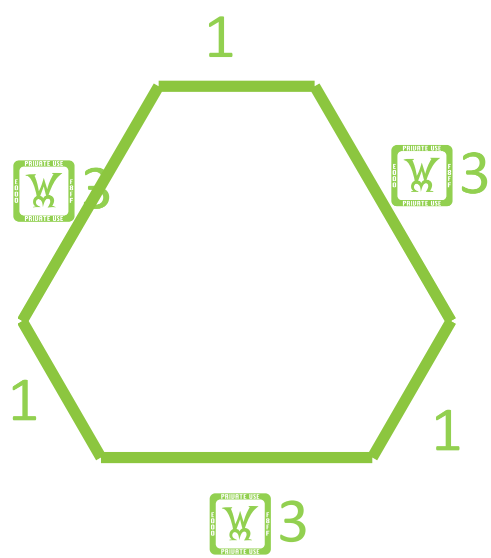


In focus: number of edges & vertices, lengths of sides

Inner Horizon: lines of symmetry, area, measure of angles

Outer Horizon: measurements can be expressed via algebraic equations – connections between strands (alg. & geo.), and between concepts (ratios of lengths, angles)

First Questionnaire



Delia's Hexagon

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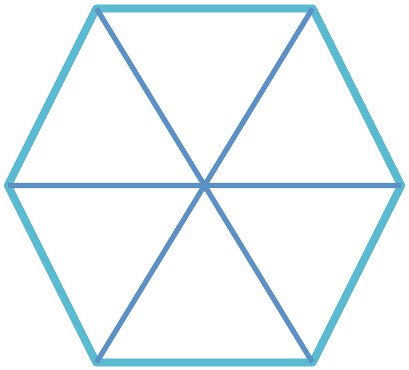
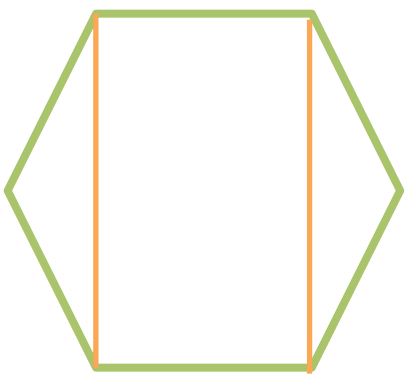
How do you recommend Delia go about finding the area?

Trends and Responses





- 18 of 20 drew diagrams, decomposed the shape – 15 were <u>regular hexagons</u>

Recommendations were

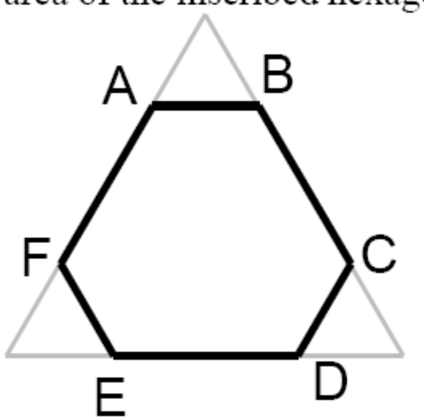
(i) based on broad ideas: “put in lines to break it up into shapes which we have established rules and laws to work with” (Sophia);

(ii)giving step-by-step procedures of how to solve: “she can solve the area of the two triangles and the rectangle in the middle using formulas ... [then[she can just add the area of the rectangle and the two triangles” (Abigail)

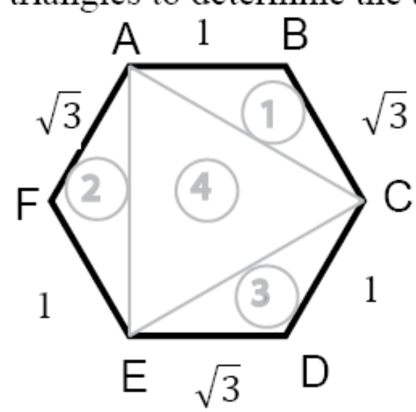
Follow up Questionnaire

To determine the area, Delia was given a variety of different recommendations. Here

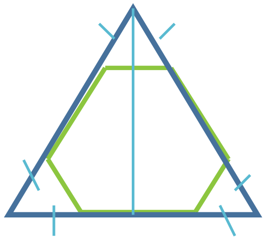
Recommendation A:
Extend the hexagon into an equilateral triangle as in the figure below. Then use the areas of the large triangle, and small outer triangles, to determine the area of the inscribed hexagon.



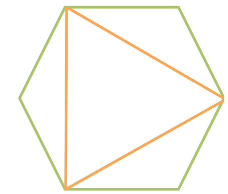
Recommendation B:
Decompose the hexagon into three triangles (1, 2, 3, which are all equal), and an equilateral triangle 4, as in the figure below. Then sum the areas of the inscribed triangles to determine the area of the hexagon.



Which approach do you prefer, and why?



Trends and Responses



“A is easier” (11/20)

Attended to structural features & consequences of the provided diagram

Sarah: “you’r

A shape’s measurements may be determined indirectly and the strategy applies beyond the specific question (Outer Horizon)

can prove very useful in fields extending beyond pure mathematics”

“B is easier” (9/20)

Attended to surface features of the solving process and prior personal experiences

Nimah: “only

Implications of the approach were not considered – e.g. how to find addends or subtrahends (Restricted KMH)

A the way I learned geometry in school... Decomposing shapes into smaller shapes”

Correlating KMH to pedagogy

Robust KMH

- able to view different aspects of the problem
- think more broadly about the student learning experience
- attention to different current and future experiences for their students

Limited KMH

- unable to accurately predict what would be challenging or easy
- generalized own thinking and comforts to students'
- these expectations for student thinking limited participants' address of the mathematics

A case for developing KMH

Watson (2008): extended experiences in mathematics at an advanced level, in terms of concepts, combining concepts, analysing mathematical statements and structures, are important aspects of teacher education that lend themselves to effective pedagogical decisions; research on such experiences is needed

Ball and Bass (2009): KMH: "engages those aspects of the mathematics that, while perhaps not contained in the curriculum, … illuminate and confer a comprehensible sense of [their] larger significance"(p. 6).

Future Directions

More in-depth research is needed:

What could be included in undergraduate mathematics, or pre-service teacher, education in order to encourage a flexible use of KMH?

In a real teaching situation, how would pre- and in-service teachers respond to this problem?

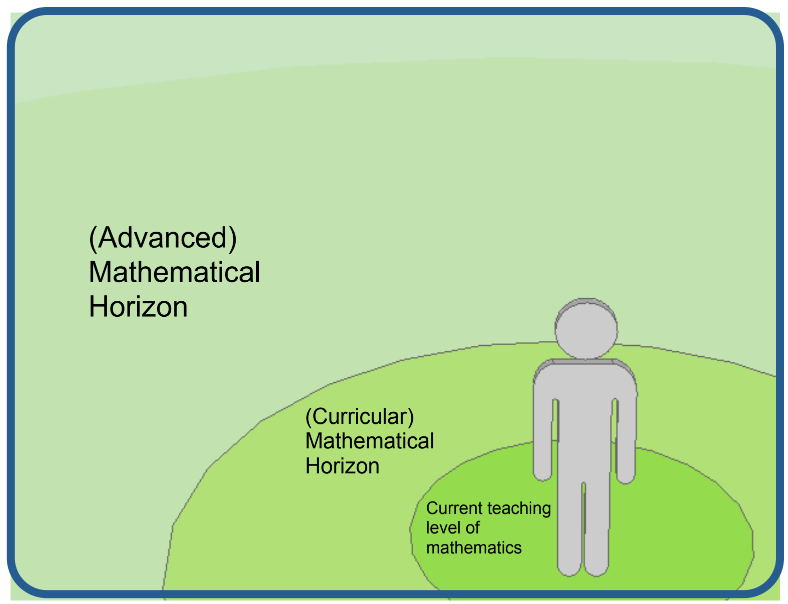
What is the connection between KMH and Knowledge of Content and Students?

HCK and Abstract Algebra

Wasserman

HCK

- Horizon Content Knowledge
  - Component of *teachers'* mathematical knowledge
  - "Mathematical horizon" indicates relative location of content, broader disciplinary territory
  - Teachers use knowledge of this broader territory to *inform instructional practices at a particular (current) location*

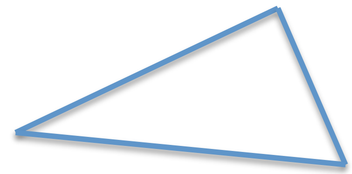


HCK & Teaching

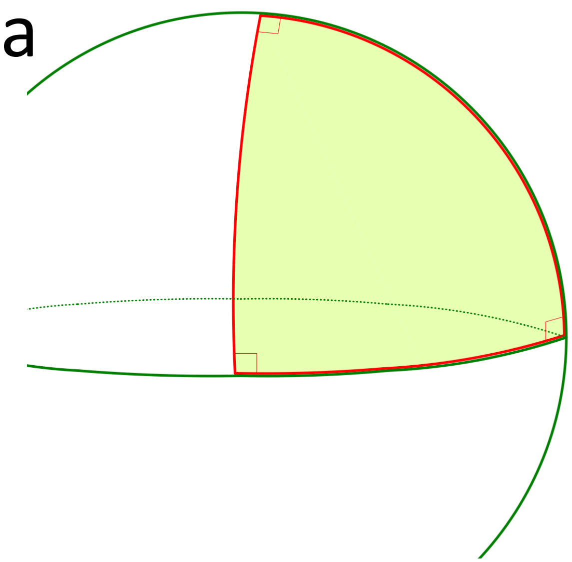
- How does this perspective and HCK alter teachers' instructional approaches and contribute to the work of teaching? What aspects of teaching? For what purposes?
  - Does it inform their choices for sequencing content?
  - Does it impact what concepts they emphasize?
  - Does it increase the cognitive demand of their lessons?
  - Does it add to ways, examples in order to engage students?
  - Does it change their perception of the content they teach?
  - Does it alter their attention to and exposition of ideas?
  - Does it shape ways they transition students to new ideas?

Use in planned practice

- Interior angle sum

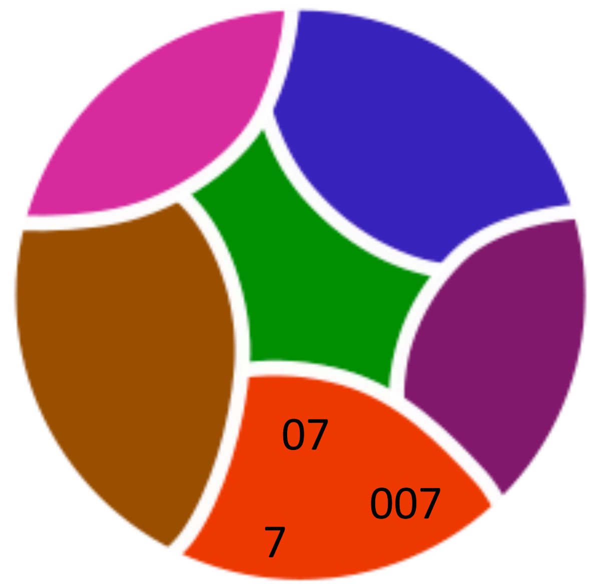


- Mr. Reese' s use of Non-Euclidean geometry to engage students (via cognitive conflict) in critical thinking and proof about planar triangles (interior angle sum) – despite students familiarity with the 180^{o} idea



(Wasserman & Stockton, 2013)

Use in planned practice



- Mrs. Smith's study of equivalence relations, and the image of them partitioning a set into equivalence classes, informed her transitioning students from N to Q+ (i.e., equivalent fractions). She discussed some of the different ways 7 is expressed – 07 (a clock 4:07), or 007 (James Bond image), adding connection and coherence for introducing the idea that the same fraction (e.g., ½) has many different forms (e.g., $^2/_4$, $^3/_6$, etc.)

Focusing on Abstract Algebra

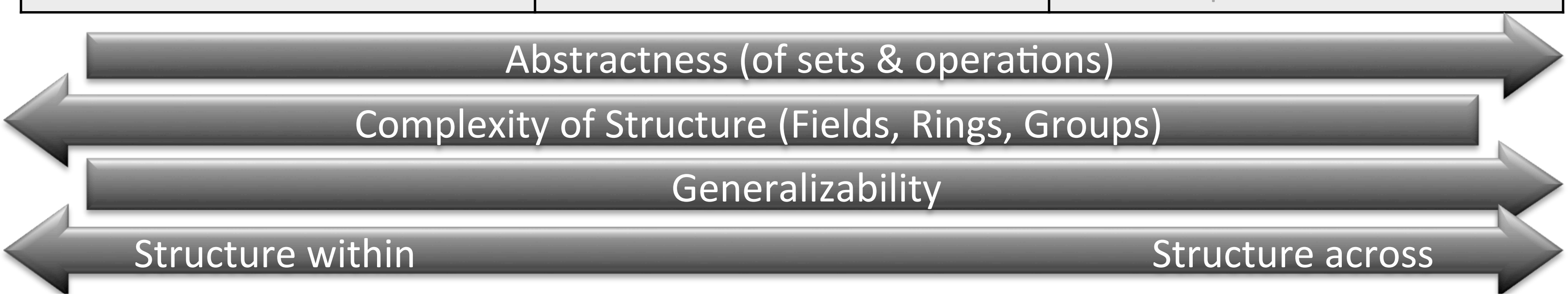
- How does teachers' knowledge of abstract algebra impact their algebra (or early algebra) teaching?
- Results from three studies
  - CCSS-M Mapping
  - Early Algebra teaching
  - Secondary teachers and inverse functions

CCSS-M Mapping

- Analyzing K-12 content (CCSS-M) that may be transformed by teachers' knowledge of abstract algebra ideas (by two researchers). Connected to broader CCSS-M analysis thru lens of teachers' knowledge of advanced mathematics.
  - 7.NS.A.2.a. Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as (–1)(–1) = 1 and the rules for multiplying signed numbers.
  - F-BF.B.4. Find inverse functions. (b) Verify by composition that one function is the inverse of another.
  - A-APR.1. Understand that polynomials form a system analogous to the integers…
- What K-12 concepts may be impacted and how?

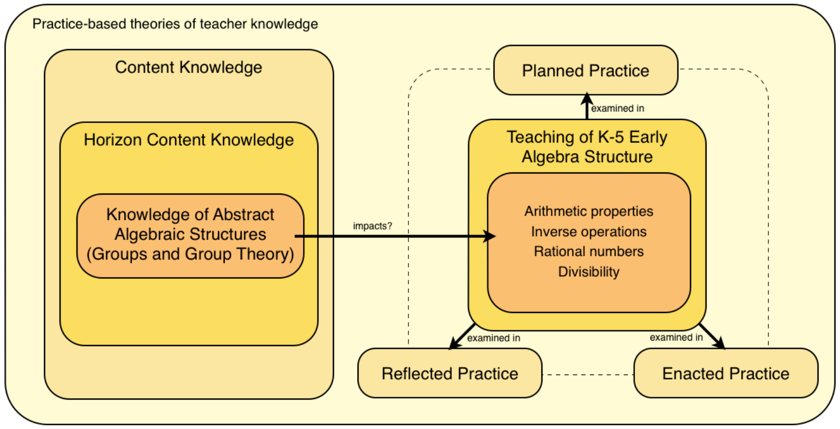
CCSS-M Mapping

Elementary School	Middle School	High School
Arithmetic Properties of Addition, Multiplication on N,Q+	of Addition, Multiplication on Z, Q; Algebraic expressions	of Addition, Multiplication on R, C; Polynomials, Functions, Transformations, Matrices
Inverses inverse operations	inverse elements (undoing equations)	inverse functions
Solving Equations (guess and check)	systematic solving, addition and multiplication	systematic solving, more complex functions; systems of equations
Structure within Sets number sets (equivalence): Q+	Z, Q; linear functions	R, C (conjugates); polynomials, rational expressions



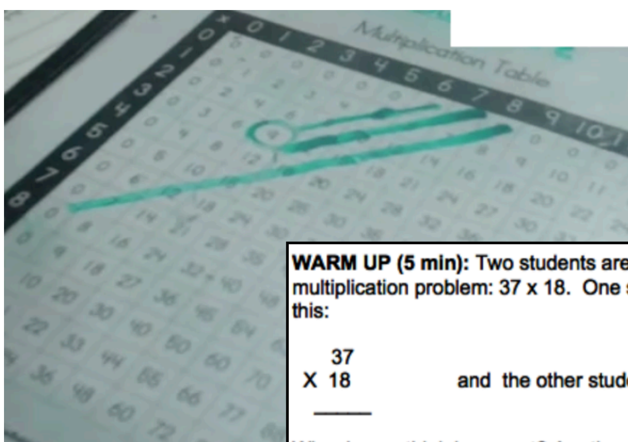
Early Algebra

- Qualitative study of 3 elementary teachers
  - Gathered pre-instruction practices on elementary topics (arithmetic properties, inverse operations, rational numbers), and post-instruction practices.
  - Teachers participated in content-focused course for in-service teachers including introduction to abstract algebra ideas, primarily groups.
  - Content typical of introductory abstract algebra; instructional approach likely different



Summary of Findings

- Initial: Lessons characterized by 1) emphasis on procedures and rules for arithmetic; and 2) avoiding or not engaging in arithmetic properties
- Post: Lessons characterized by 1) purposeful use and development of arithmetic properties; 2) turning instruction about early structure in meaningful exercises in sense-making; and 3) emphasis on bridging ideas and concepts for students



Teacher: Let's try another one. Let's put 1,002+0=____. What does it equal?
Students: 1,002.
Teacher: Did you need your boards for that?
Students: No!
Teacher: Why not?
Student (one of many hands up): Because every number added to 0 will stay the same.
Teacher: Identity property. Identity is you, isn't it? Your identity is who you are. 6 is 6 isn't it? We add 0 to it, it's still 6, it still has the same identity. So it kind of goes along with that

a number for multiplication that does that. Can you think of the number that no matter what you multiply the number stays the same?

ave the identity property for addition is 0, and the identity property for . You've been using it already this year, you just didn't know what it was ou see it today, you'll go, 'I know what that is.'

WARM UP (5 min): Two students are working on the following multiplication problem: 37 x 18. One student arranges the problem like this:

$$\begin{array}{r} 37 \\ \times\ 18 \\ \hline \end{array} \quad \text{and the other student does this:} \quad \begin{array}{r} 18 \\ \times\ 37 \\ \hline \end{array}$$

Who do you think is correct? Are the products of these two problems different? Solve and share with your pair.

Summary of Findings

- *"Well for me, the arithmetic properties were just a set of useless statements that students did not need to understand. However, when doing the activity with the [group of triangle symmetries], I started to see that the arithmetic properties were more than just unpractical statements… Then, as I prepared one of the assigned lessons on the multiplication chart, I realized students would benefit a lot if they understood arithmetic properties before introducing 3-digit multiplication."*

Summary of Findings

- Changes in instruction appear related to knowledge of mathematical horizon
  - More *aware* of arithmetic structure in elementary setting, related to grappling with these ideas in abstract contexts
  - More *connected* understanding of structure, related to collective aspects of properties in algebraic structures
  - More *purposeful* with arithmetic structure, related to appreciation of broader disciplinary landscape

Secondary Inverse Functions

- Task-based interviews with middle/secondary teachers
  - Recently finished graduate level course in Abstract Algebra, stratified sample based on course performance
  - Mathematical tasks and interview questions probed, amongst other aims, whether teachers connected to and used the group of functions under composition to explain inverse functions and/or solutions of more complex equations

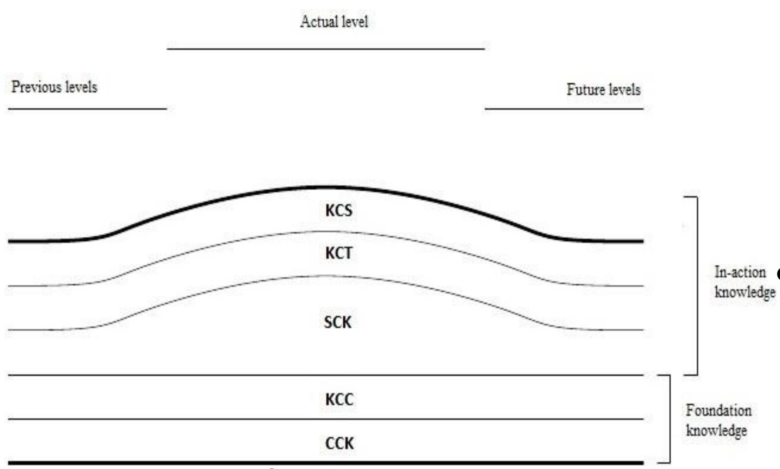
Questions for Consideration

- What data sources may best capture HCK and its impact on teaching?
- What distinctive features of teachers' knowledge of content best capture HCK?
- Others?

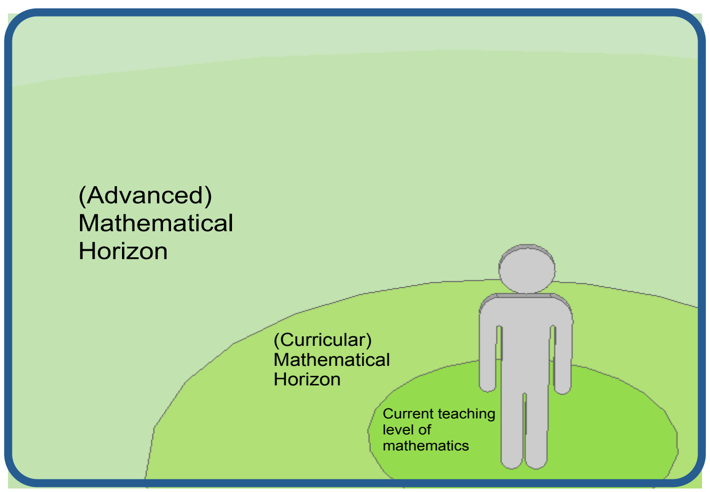
Across the Three Presentations

What are your reflections?

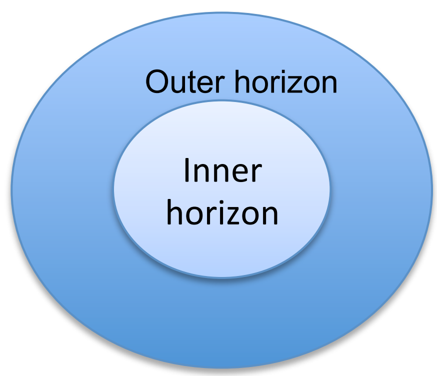
Group Discussion

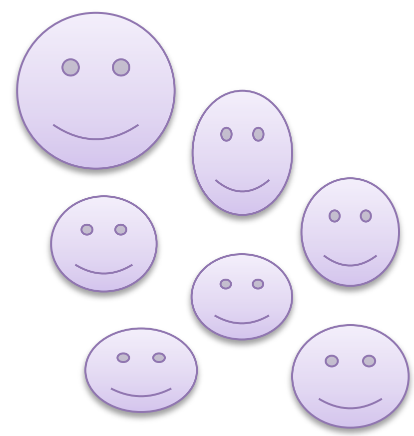


Fernandex & Figueiras, 2014



- In what ways can HCK / KMH support or enhance other facets of teachers' disciplinary knowledge? (and vice versa)
- How may these different conceptualisations of HCK / KMH support or enhance one another? (in research and practice)
- What questions / issues / concerns arise for you?





Themes for Day 2:

- What primary impacts might HCK have on the work of teaching? What are some examples / episodes from your own classroom experiences?

- What are some methodological approaches (potentialities and constraints) to study?

- What are some ways to access and develop HCK in and for teaching?

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